Caesar: A Deductive Verifier for Probabilistic Programs

Dafny 2024

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The *geometric loop* program C_{geo}:

$$
\begin{array}{l}
c := 0;\ run := \texttt{true}; \\
\texttt{while}\,(run)\,\{ \\
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4. Expected runtime?

   2.0 ("positively almost-surely terminating")

Our Verification Infrastructure for Probabilistic Programs

lower and upper bounds on expected values

(amortized) expected run-times

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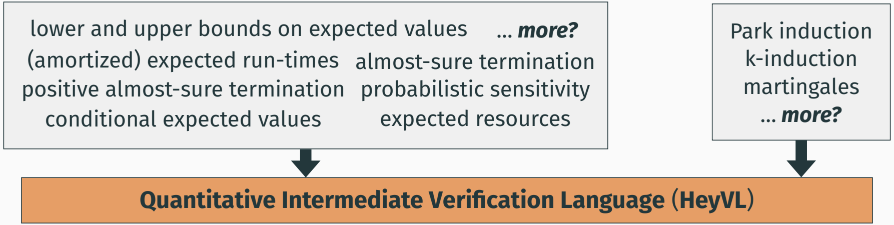
⇒ An intermediate language for the verification of probabilistic programs.
"Build the probabilistic version of *Boogie/Viper*"

Our Verification Infrastructure for Probabilistic Programs



We present:

- A novel *intermediate language* to verify probabilistic programs (*HeyVL*),

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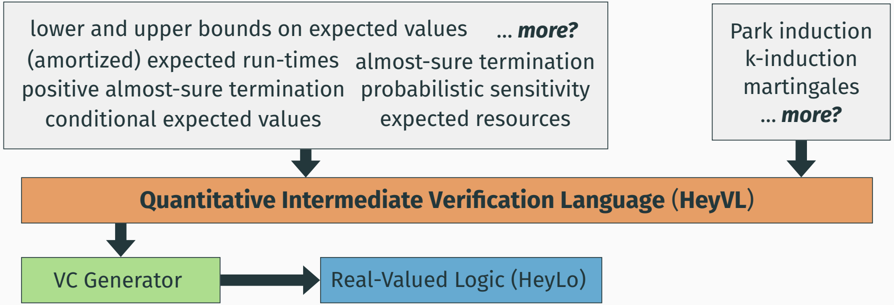

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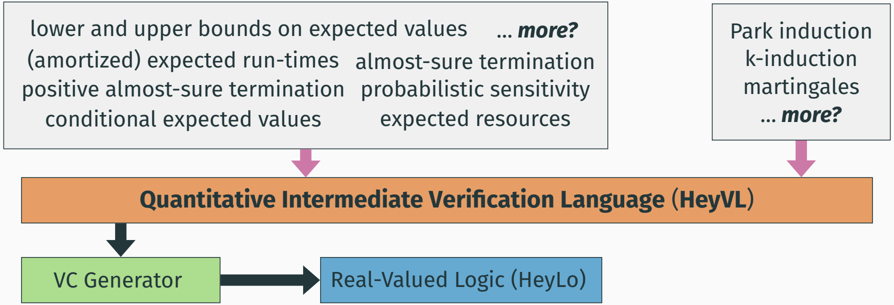

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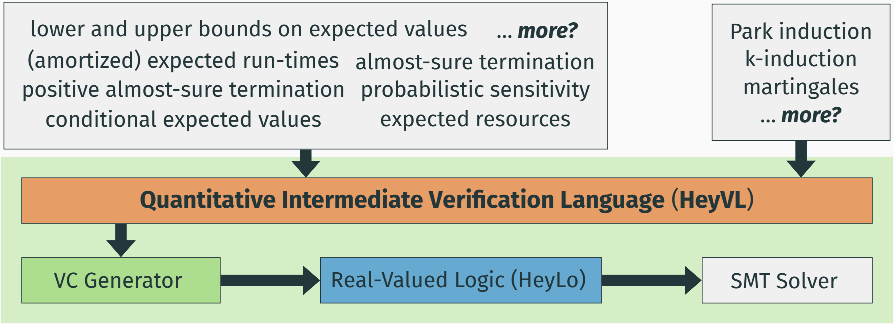

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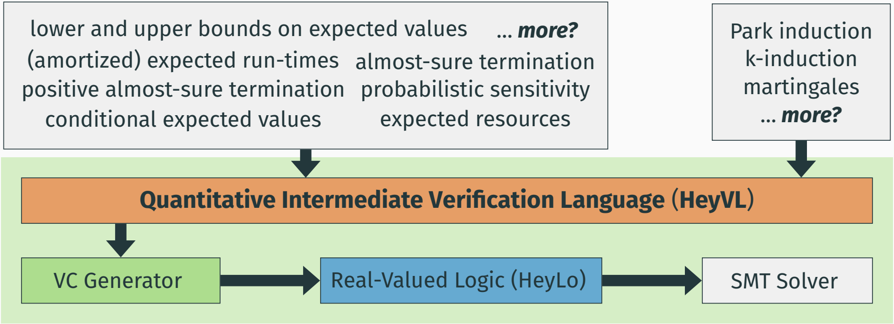

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	Predicates	*Expectations*
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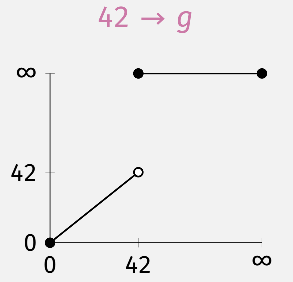

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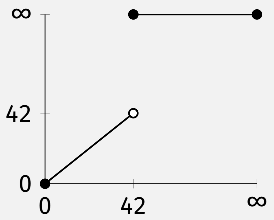
$42 \rightarrow g$



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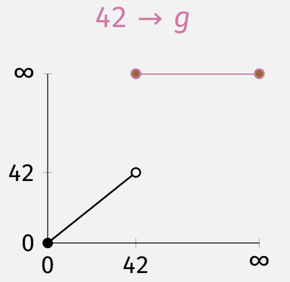

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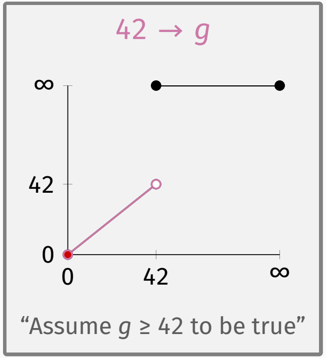

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assume c + 0.5
{ run := false } [0.5] { c := c + 1 }
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The HeyVL program verifies, therefore

$$\{c + 0.5\}_{\leq}\ S'\ \{c\}$$

where $S' = \{\, run := \texttt{false} \,\}\, [0.5]\, \{\, c := c + 1 \,\}$.

HeyVL

Expectations

lower bounds

S	$[\![S]\!](g)$
`assert` f	$f \sqcap g$
`assume` f	$f \to g$
`havoc` x	$\inf x\colon g$

Omitted: `validate`, `reward`, branching

Classical IVL

Predicates

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`assert` P	$P \wedge Q$
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upper bounds

S	$[\![S]\!](g)$
`coassert` f	$f \sqcup g$
`coassume` f	$f \leftarrow g$
`cohavoc` x	$\sup x \colon g$

Omitted: `validate`, `reward`, branching

- **HeyVL** generalizes classical IVLs.
- **HeyVL** has dual verification statements for *upper bounds reasoning*.

More than 40 examples:

- with 12 proof rules for loops,
- lower and upper bounds,
- procedures with recursion,
- user-defined data structures.

Case Studies

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Problem	Verification Technique	Source
LPROB	wlp + Park induction	McIver and Morgan [2005]
	wlp + latticed k-induction	**(new?)**
UPROB	wlp + ω-invariants	Kaminski [2019]
LEXP	wp + ω-invariants	Kaminski [2019]
	wp + Optional Stopping Theorem	Hark et al. [2019]
UEXP	wp + Park induction	McIver and Morgan [2005]
	wp + latticed k-induction	Batz et al. [2021]
CEXP	conditional wp	Olmedo et al. [2018]
LERT	ert calculus + ω-invariants	Kaminski et al. [2016]
UERT	ert calculus + UEXP rules	Kaminski et al. [2016]
AST	parametric super-martingale rule	McIver et al. [2018]
PAST	program analysis with martingales	Chakarov and Sankaranarayanan [2013]
???	**more proof rules**	**you?**

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HeyVL

Expectations

Let $I \in \mathbb{E}$ be an invariant candidate.

```
assert I
havoc x⃗
validate; assume I
if (b) {
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```
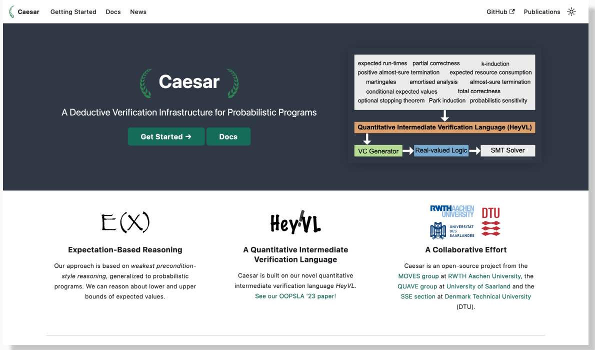

In our OOPSLA ’23 paper:

- *HeyVL:* An *intermediate language* to verify probabilistic programs,
- *HeyLo:* An *assertion language* to reason about quantities,
- Case studies of HeyVL encodings.

Online:

- The verifier Caesar
  - written in Rust, open source
- Language documentation
- Extended version of the paper



`www.caesarverifier.org`